How to Extrapolate A String Model to Finite Temperature: Interpolations and Implications for the Hagedorn Transition

Keith R. Dienes[*], Michael Lennek[†]

Department of Physics, University of Arizona, Tucson, AZ 85721 USA

(Dated: July 20, 2005)

In this paper, we discuss the important question of how to extrapolate a given zero-temperature string model to finite temperature. It turns out that this issue is surprisingly subtle, and we show that many of the standard results require modification. For concreteness, we focus on the case of the ten-dimensional $SO(32)$ heterotic string, and show that the usual finite-temperature extrapolation for this string is inconsistent at the level of a proper worldsheet theory. We then derive the proper extrapolation, and in the process uncover a universal Hagedorn temperature for all tachyon-free closed string theories in ten dimensions — both Type II and heterotic. As we discuss, these results are not in conflict with the well-known exponential growth in the degeneracies of string states in such models. This writeup is a concise summary of our recent paper `hep-th/0505233`, here presented using a "bottom-up" approach based on determining self-consistent finite-temperature extrapolations of zero-temperature string models. Some new results and observations are also added.

I.   PRELIMINARIES: THERMAL ORBIFOLDS AND PARTITION FUNCTIONS

Our first task is to understand how to extrapolate a given zero-temperature theory to finite temperature. In this section, we shall quickly review the standard procedures for doing this.

Within the language of quantum field theory, there is a well-known procedure for extrapolating a given zero-temperature model to finite temperature T: we compactify a Euclidean time-like dimension on a circle of radius $R_T = (2\pi T)^{-1}$, and require that bosonic (fermionic) fields be periodic (anti-periodic) around this circle. The thermal Matsubara modes are then nothing but the Kaluza-Klein states corresponding to this compactification, and the periodic (anti-periodic) boundary conditions for bosons (fermions) are imposed in order to ensure proper thermal spin-statistics relations.

This basic picture also holds in string theory [1, 2]: again we compactify a Euclidean time-like dimension on a circle of radius R_T, obtaining a theory with one fewer spacetime dimension, and again we demand appropriate boundary conditions for spacetime bosons/fermions around the thermal circle. However, for closed strings, there is a new feature: in addition to Kaluza-Klein (Matsubara) momentum states, there are also Matsubara winding states which arise from closed strings wrapping around this thermal circle. Such states are needed for modular invariance, and encourage us to think of this thermal circle quite literally as a compactified spacetime dimension.

It will be important for us to see how this works at the level of explicit string model-building and the associated thermal partition functions. The following discussion follows the mathematical treatment in Ref. [3], suitably T-dualized in order to apply to temperature rather than geometric radius [4]. Let us suppose that we begin with a D-dimensional zero-temperature closed string model whose one-loop partition function is given by $Z(\tau)$, where τ is the complex toroidal modular parameter. The first step in the thermal construction is to compactify this theory on circle of radius R_T. At this stage, we then have a thermal string partition function $Z_{\text{therm}}(\tau, T)$ of the form

$$Z_{\text{therm}}(\tau, T) \;\equiv\; Z(\tau) \, Z_{\text{circ}}(\tau, T) \tag{1}$$

where the extra factor Z_{circ} represents a double summation over integer Matsubara momentum and winding modes:

$$Z_{\text{circ}}(\tau, T) \;=\; \sqrt{\tau_2} \sum_{m,n \in \mathbb{Z}} \overline{q}^{\,(ma - n/a)^2/4} \, q^{(ma + n/a)^2/4} \tag{2}$$

with $a \equiv 2\pi T/M_{\text{string}}$ and $\tau_2 \equiv \text{Im}\,\tau$. Here (m, n) represent the thermal momentum and winding numbers, respectively. However, at this stage in the construction, we see that each of the states within $Z(\tau)$ is multiplied by the same thermal

---

[*] E-mail address: `dienes@physics.arizona.edu`
[†] E-mail address: `mlennek@physics.arizona.edu`

spectrum of integer momentum and winding modes within Z_{circ}. The next step, therefore, is to break this degeneracy, ensuring that while bosonic states within $Z(\tau)$ continue to have integer Matsubara modes, fermionic states should have *half-integer* modings (so that they are anti-periodic around the thermal circle).

In string theory, the only way to accomplish this in a self-consistent manner is by twisting or orbifolding the compactified theory in Eq. (1). What orbifold do we choose? Clearly, we need a $\mathbb{Z}_2$ operator that distinguishes between spacetime bosonic and fermionic states, such as $(-1)^F$ (where F represents spacetime fermion number). We shall generally let Q denote such an operator, since we shall eventually argue that Q must contain more than merely $(-1)^F$. However, we will also need to couple Q with an operator that can distinguish between between integer and half-integer thermal momenta. As we shall see, such an operator is given by $\mathcal{T} : y \to y + \pi R_T$, where y is the (T-dual) coordinate along the compactified dimension. This is nothing but a shift around half the circumference of the (dualized) thermal circle, so that the states which are invariant under $\mathcal{T}$ are those with even winding numbers. This will then necessarily re-introduce states with odd winding numbers in the twisted sectors, along with states having half-integer momentum numbers.

Given these operators, the final step in our procedure is to orbifold the circle-compactified theory in Eq. (1) by the $\mathbb{Z}_2$ product operator $\mathcal{T}Q$. What does this do to our partition function? While Q acts on the original non-thermal component $Z(\tau)$, the operator $\mathcal{T}$ acts on the thermal sum $Z_{\text{circ}}(\tau, T)$. Since states contributing to Z_{circ} with even (odd) values of n are even (odd) under $\mathcal{T}$, let us distinguish the specific values of m and n by introducing [3] four new thermal functions $\mathcal{E}_{0,1/2}$ and $\mathcal{O}_{0,1/2}$ which are the same as the summation in Z_{circ} in Eq. (2) except for the following restrictions on their summation variables:

$$
\begin{aligned}
\mathcal{E}_0 &= \{m \in \mathbb{Z},\ n \text{ even}\} \\
\mathcal{O}_0 &= \{m \in \mathbb{Z},\ n \text{ odd}\} \\
\mathcal{E}_{1/2} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ n \text{ even}\} \\
\mathcal{O}_{1/2} &= \{m \in \mathbb{Z} + \tfrac{1}{2},\ n \text{ odd}\} \ .
\end{aligned}
\tag{3}
$$

Note that $Z_{\text{circ}} = \mathcal{E}_0 + \mathcal{O}_0$. Given this, our original (untwisted) thermal partition function in Eq. (1) can be rewritten as

$$
Z_{\text{therm},+}^+ = Z_+^+\ (\mathcal{E}_0 + \mathcal{O}_0)
\tag{4}
$$

where $Z_+^+(\tau) \equiv Z(\tau)$. Therefore, in order to project onto the states invariant under $\mathcal{T}Q$, we add to Eq. (4) the contributions from the projection sector

$$
Z_{\text{therm},+}^- = Z_+^-\ (\mathcal{E}_0 - \mathcal{O}_0)
\tag{5}
$$

where Z_+^- is the Q-projection sector for the non-thermal contribution Z_+^+. In the usual fashion, modular invariance then requires us to add the contribution from the twisted sector

$$
Z_{\text{therm},-}^+ = Z_-^+\ (\mathcal{E}_{1/2} + \mathcal{O}_{1/2})
\tag{6}
$$

as well as its corresponding projection sector

$$
Z_{\text{therm},-}^- = Z_-^-\ (\mathcal{E}_{1/2} - \mathcal{O}_{1/2}) \ .
\tag{7}
$$

The net result of the orbifold, then, is a $(D-1)$-dimensional thermal string model with total partition function

$$
\begin{aligned}
Z_{\text{string}}(\tau, T) &= \frac{1}{2}\left(Z_{\text{therm},+}^+ + Z_{\text{therm},+}^- + Z_{\text{therm},-}^+ + Z_{\text{therm},-}^- \right) \\
&= \frac{1}{2}\left\{ \mathcal{E}_0\left(Z_+^+ + Z_+^-\right) + \mathcal{E}_{1/2}\left(Z_-^+ + Z_-^-\right) + \mathcal{O}_0\left(Z_+^+ - Z_+^-\right) + \mathcal{O}_{1/2}\left(Z_-^+ - Z_-^-\right) \right\} \ .
\end{aligned}
\tag{8}
$$

It is straightforward to interpret the physics of this thermal model. As $T \to \infty$, we find that $\mathcal{E}_{1/2}$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0$ and $\mathcal{O}_0$ become equal; thus the partition function of our thermal model reduces to

$$
Z_{\text{string}}(\tau, T) \ \to \ Z_+^+ \equiv Z \qquad \text{as } T \to \infty \ .
\tag{9}
$$

In other words, we see that the original D-dimensional model with which we started can now be interpreted as the $T \to \infty$ limit of the $(D-1)$-dimensional thermal model we have constructed. By contrast, as $T \to 0$, we find that $\mathcal{O}_0$ and $\mathcal{O}_{1/2}$ each vanish while $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ become equal. Thus

$$
Z_{\text{string}}(\tau, T) \ \to \ \tfrac{1}{2}\left(Z_+^+ + Z_+^- + Z_-^+ + Z_-^- \right) \qquad \text{as } T \to 0 \ .
\tag{10}
$$

However, this is nothing but the Q-orbifold of the original D-dimensional model with which we began. Of course, since Q is a $\mathbb{Z}_2$ operator, we know that $Q^2 = \mathbf{1}$. We may therefore change our perspective and equivalently view our D-dimensional $T \to \infty$ model as the Q-orbifold of our D-dimensional $T \to 0$ model.

Thus, to summarize, we see that all finite-temperature string models must have partition functions of the modular-invariant form [3, 4]

$$Z_{\text{string}}(\tau, T) \; = \; Z^{(1)}(\tau)\, \mathcal{E}_0(\tau, T) \; + \; Z^{(2)}(\tau)\, \mathcal{E}_{1/2}(\tau, T) \; + \; Z^{(3)}(\tau)\, \mathcal{O}_0(\tau, T) \; + \; Z^{(4)}(\tau)\, \mathcal{O}_{1/2}(\tau, T) \; , \qquad (11)$$

where $Z^{(i)}$ represent general, model-specific, non-thermal contributions to the total thermal partition function Z_{string}. In the $T \to 0$ limit, we obtain a partition function of the form

$$Z_{\text{model}} \; = \; Z^{(1)} + Z^{(2)} \; , \qquad (12)$$

and thus we may interpret Eq. (11) as describing the finite-temperature extrapolation of the zero-temperature model described in Eq. (12). The fact that the orbifold Q contains a $(-1)^F$ factor guarantees that finite-temperature effects will break whatever supersymmetry might have existed at zero temperature. By contrast, the opposite $T \to \infty$ limit yields

$$\tilde{Z}_{\text{model}} \; = \; Z^{(1)} + Z^{(3)} \; , \qquad (13)$$

which corresponds to a different D-dimensional string model. Thus, the thermal partition function in Eq. (11) can be viewed as mathematically *interpolating* between one zero-temperature string model at $T = 0$ [whose partition function is given in Eq. (12)] and a *different* zero-temperature string model as $T \to \infty$ [whose partition function is given in Eq. (13)]. These two models are related directly in D dimensions through the action of the $\mathbb{Z}_2$ orbifold operator Q.

This is a general result, so it bears repeating: *All D-dimensional thermal models are $(D-1)$- dimensional interpolating models, with the temperature T serving as an interpolating parameter. As $T \to 0$, we obtain a D-dimensional string model M_1; this is identified as the zero-temperature string model whose thermal extrapolation we have constructed. By contrast, as $T \to \infty$, we obtain a different D-dimensional string model M_2 which must be a $\mathbb{Z}_2$ orbifold of M_1.*

A comment on semantics is in order here. Strictly speaking, in the $T \to \infty$ limit we obtain a $(D-1)$-dimensional degenerate (*i.e.*, zero-radius) model M_2 which is actually only T-dual to a D-dimensional model. Thus, if M_2 is the $T \to \infty$ limit of our $(D-1)$-dimensional thermal interpolating model, then we should more correctly state that our $(D-1)$-dimensional thermal model interpolates between the D-dimensional models M_1 and $\tilde{M}_2$, where $\tilde{M}_2$ is the T-dual of M_2. In some sense, this distinction is only a matter of semantics, having to do with the naming of the $T \to \infty$ endpoint of the interpolation; moreover, for closed strings we should properly regard both M_2 and $\tilde{M}_2$ as being D-dimensional since they each have a continuous spectrum of states associated with the formerly compactified dimension. For simplicity, therefore, we shall continue to refer to such an interpolating model as connecting M_1 and M_2 in the remainder of this paper. However, it is important to note that it is M_2 (and not $\tilde{M}_2$) which must be the Q-orbifold of M_1.

Note that for each specified D-dimensional zero-temperature string model M_1, there will in general exist many $(D-1)$-dimensional string models which extrapolate away from it. This depends on the choice of the second model M_2 to which one interpolates, or equivalently on the choice of the $\mathbb{Z}_2$ orbifold Q. In other words, the requirement of modular invariance alone is not sufficient to determine a unique interpolation, and is therefore not sufficient to determine a unique thermal extrapolation. The next step, then, is to determine a set of criteria for selecting the appropriate orbifold Q.

II. WHAT DEFINES A SELF-CONSISTENT FINITE-TEMPERATURE EXTRAPOLATION?

Let us now enumerate what we believe are the weakest possible conditions that can be imposed in order to have a self-consistent finite-temperature extrapolation of a given D-dimensional zero-temperature string model. Throughout, our goal is to impose only the most conservative requirements for self-consistency. The conditions we shall impose are as follows:

- First, the finite-temperature extrapolation should represent a valid, self-consistent $(D-1)$-dimensional string model in its own right. In other words, it should satisfy all necessary worldsheet constraints such as conformal and superconformal invariance, self-consistent GSO projections, zero-temperature spin-statistics relations, *etc*.

- Second, this model should have an identifiable radius modulus corresponding to a *bona-fide* geometric compactification circle.

- Finally, this compactification circle should be interpretable *thermally* in the field-theory limit. This means that all states which survive in the field-theory limit should satisfy field-theoretic *thermal* spin-statistics relations.

In practice, this last requirement means that all massless spacetime bosons (fermions) with zero thermal windings in the theory should have thermal momentum excitations which are periodic (anti-periodic) around the thermal circle. Note, in particular, that we do *not* make any demands on the states with non-zero thermal winding modes; such stringy states have no field-theoretic limits, and are beyond our usual expectations. Likewise, by restricting our attention to only the *massless* states, we are again focusing on only the light states which can emerge in an appropriate low-energy field-theoretic limit. We stress that such *thermal* spin-statistics relations should be contrasted with the *zero-temperature* spin-statistics relations mentioned in the first of our requirements, which demand only that spacetime bosons (fermions) contribute with a positive (negative) sign to the overall partition function.

Given these conditions, we can now examine how well our construction in Sect. I fares. Clearly, our first condition is automatically satisfied for *any* orbifold Q, since the construction of our thermal model in Sect. I proceeded by legitimate string-theoretic steps such as compactification and orbifolding. As long as our original zero-temperature model is self-consistent and as long as the orbifold Q is a legitimate allowed orbifold for this model, we are guaranteed that the resulting thermal model satisfies the first constraint. Similarly, the second constraint is also satisfied, again by construction.

However, the third constraint is more subtle. At first glance, it might seem that we have also satisfied our third constraint when we assumed that Q contains a $(-1)^F$ factor and coupled it with the half-shift $\mathcal{T}$ when constructing our thermal orbifold. However, there are two reasons why this may fail to be the case. First, our orbifold Q may generally contain other factors beyond $(-1)^F$ (such as gauge-group Wilson lines). Thus, in such cases, the thermal circle periodicities would be correlated not with spacetime fermion number alone, but with a combination of spacetime fermion number and Wilson-line eigenvalues. Of course, one might attempt to avoid this by taking $Q = (-1)^F$ directly, with no additional Wilson-line factors. In cases when this can be done, this issue will not arise, but we shall see shortly that this cannot always be done. The second reason, however, is more general. Recall that in our construction in Sect. I, we began with a D-dimensional string model which turned out to be the $T \to \infty$ limit of the thermal model we eventually constructed. Indeed, it was only the Q-orbifold of this model which emerged in the zero-temperature limit. Therefore, when we implemented the orbifold factor $(-1)^F$ in our construction, this acted on the bosons/fermions of the $T \to \infty$ model, but not necessarily those of the $T \to 0$ model. In other words, if a model interpolating from M_1 to M_2 is the proper thermal extrapolation for the zero-temperature model M_1 (obeying proper thermal spin-statistics relations for the bosons and fermions of M_1), there is no guarantee that the T-dual model, which interpolates from M_2 to M_1, will be the appropriate thermal extrapolation for the model M_2. Thus, the construction we outlined in Sect. I — although completely general — is not by itself capable of guaranteeing that we have successfully maintained proper thermal spin-statistics relations when the final $(D-1)$-dimensional interpolation is constructed. Note that this remains true even in cases when $Q = (-1)^F$.

It is therefore this third requirement involving proper thermal spin-statistics relations which can be used to select the proper orbifold Q, and with it the correct thermal extrapolation for a given string model. We shall give two explicit examples of this procedure below.

III.   RELATION TO THE STANDARD APPROACH

Let us now compare how our discussion thus far relates to the standard prescription in the literature [4] for constructing the thermal extrapolation of a given zero-temperature string model. We shall explicitly work out two examples in ten dimensions: the supersymmetric Type IIA/IIB superstrings, and the supersymmetric $SO(32)$ heterotic string. As we shall show, the standard prescription agrees completely with the our construction in the case of the Type II superstrings, but differs in the case of the $SO(32)$ heterotic string. In the latter case, we shall explicitly describe exactly where we believe the problem lies with the usual prescription.

A.   The Type II strings

Let us begin by considering the case of the ten-dimensional Type II superstrings. For concreteness, we shall focus on the (chiral) Type IIB string; the case of the (non-chiral) Type IIA string proceeds in exactly the same manner. The Type IIB string at zero temperature has the partition function

$$Z_{\mathrm{IIB}} \;=\; Z_{\mathrm{boson}}^{(8)} \, (\overline{\chi}_V - \overline{\chi}_S)\,(\chi_V - \chi_S) \tag{14}$$

where the contribution from the worldsheet bosons is given in terms of the Dedekind η-function as

$$Z_{\text{boson}}^{(n)} \; \equiv \; \tau_2^{-n/2} \, (\overline{\eta}\eta)^{-n} \; , \tag{15}$$

and where the contributions from the left–moving (right-moving) worldsheet fermions are written in terms of the unbarred (barred) characters χ_i ($\overline{\chi}_i$) of the transverse $SO(8)$ Lorentz group. In general, the subscripts I, V, S, and C refer to the identify, vector, spinor, and conjugate spinor representations of any $SO(2n)$ affine Lie group; these representations have conformal dimensions $\{h_I, h_V, h_S, h_C\} = \{0, 1/2, n/8, n/8\}$, and have corresponding characters which can be expressed in terms of Jacobi ϑ-functions as

$$
\begin{aligned}
\chi_I &= \tfrac{1}{2} \left(\vartheta_3{}^n + \vartheta_4{}^n\right)/\eta^n &= q^{h_I - c/24} \left(1 + n(2n-1)\, q + ...\right) \\
\chi_V &= \tfrac{1}{2} \left(\vartheta_3{}^n - \vartheta_4{}^n\right)/\eta^n &= q^{h_V - c/24} \left(2n + ...\right) \\
\chi_S &= \tfrac{1}{2} \left(\vartheta_2{}^n + \vartheta_1{}^n\right)/\eta^n &= q^{h_S - c/24} \left(2^{n-1} + ...\right) \\
\chi_C &= \tfrac{1}{2} \left(\vartheta_2{}^n - \vartheta_1{}^n\right)/\eta^n &= q^{h_C - c/24} \left(2^{n-1} + ...\right)
\end{aligned}
\tag{16}
$$

where the central charge is $c = n$ at affine level one. For the ten-dimensional transverse Lorentz group $SO(8)$, the distinction between S and C is equivalent to relative spacetime chirality. Note that the $SO(8)$ transverse Lorentz group has a triality symmetry under which the vector and spinor representations are indistinguishable. Thus $\chi_V = \chi_S$ and $\overline{\chi}_V = \overline{\chi}_S$, resulting in a (vanishing) supersymmetric partition function in Eq. (14). The presence of two such factors in Eq. (14) reflects the $\mathcal{N} = 2$ supersymmetry of this model at zero temperature.

Let us now consider the extrapolation of this theory to finite temperature. If we apply the the standard prescription in Ref. [4], we obtain a nine-dimensional extrapolation with partition function

$$
\begin{aligned}
Z_{\text{string}}(\tau, T) \; = \; Z_{\text{boson}}^{(7)} \times \{ \quad &\mathcal{E}_0 \quad [\overline{\chi}_V \chi_V + \overline{\chi}_S \chi_S] \\
- \;&\mathcal{E}_{1/2} \quad [\overline{\chi}_V \chi_S + \overline{\chi}_S \chi_V] \\
+ \;&\mathcal{O}_0 \quad [\overline{\chi}_I \chi_I + \overline{\chi}_C \chi_C] \\
- \;&\mathcal{O}_{1/2} \quad [\overline{\chi}_I \chi_C + \overline{\chi}_C \chi_I] \quad \} \; .
\end{aligned}
\tag{17}
$$

It is easy to interpret this partition function in the language of our construction in Sect. I, and in the process verify that this result is self-consistent. First, we see that the $T \to 0$ limit of this expression reproduces the Type IIB partition function in Eq. (14), while the $T \to \infty$ limit of this expression yields the partition function of the non-supersymmetric ten-dimensional Type 0B superstring:

$$Z_{\text{0B}} \; = \; Z_{\text{boson}}^{(8)} \, \left(\overline{\chi}_I \chi_I + \overline{\chi}_V \chi_V + \overline{\chi}_S \chi_S + \overline{\chi}_C \chi_C\right) \; . \tag{18}$$

Thus, the nine-dimensional thermal model in Eq. (17) interpolates between the Type IIB string and the Type 0B string, thereby breaking supersymmetry for all $T > 0$. (The extra factor of $Z_{\text{boson}}^{(1)}$ required in these limits emerges as the limit of the thermal $\mathcal{E}/\mathcal{O}$ functions.) Note, moreover, that the Type 0B string is nothing but a $\mathbb{Z}_2$ orbifold of the Type IIB string, where the orbifold action is simply $Q = (-1)^F$ where F denotes the total spacetime fermion number. This is indeed a legitimate orbifold action for the Type IIB string, and is in fact the only such orbifold that would have been possible for the Type IIB theory. Finally, we note that this interpolation also satisfies proper thermal spin-statistics relations in the field-theoretic limit. Specifically, massless spacetime bosons (such as those arising within the sectors $\overline{\chi}_V \chi_V$ or $\overline{\chi}_S \chi_S$) have integer modings around the thermal circle (*i.e.*, they are multiplied by $\mathcal{E}_0$), while massless spacetime fermions (such as within $\overline{\chi}_V \chi_S$ or $\overline{\chi}_S \chi_V$) have half-integer modings around the thermal circle (*i.e.*, they are multiplied by $\mathcal{E}_{1/2}$). As discussed in Sect. II, we do not impose any requirements on massive states or states with non-zero windings (such as those corresponding to the $\mathcal{O}_0$ or $\mathcal{O}_{1/2}$ sectors), but we observe that the usual thermal spin-statistics relations actually hold in those sectors as well.

We conclude, then, that the standard prescription agrees completely with our construction for thermal string models in the case of the Type IIB string. The case of the Type IIA string is almost exactly the same; we can simply replace $\chi_S \leftrightarrow \chi_C$ for the left-moving characters throughout the above expressions. The Type IIA thermal extrapolation is therefore one which interpolates between the Type IIA string at $T = 0$ and the Type 0A string as $T \to \infty$.

Note that in each case, these extrapolations are tachyon-free up to a certain critical temperature. As we shall discuss below, their failure to remain tachyon-free beyond this temperature is the signal for the usual Type II Hagedorn transition.

B. The heterotic theory

Let us now turn to the case of the ten-dimensional $SO(32)$ heterotic string. It is here that we shall find an important difference relative to the usual thermal prescription. The ten-dimensional $SO(32)$ heterotic string has the zero-temperature partition function

$$Z_{SO(32)} \;=\; Z^{(8)}_{\rm boson} \, (\overline{\chi}_V - \overline{\chi}_S) \, (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \;. \tag{19}$$

As with the Type II string, the contribution from the worldsheet bosons is given in Eq. (15) and the contributions from the right-moving worldsheet fermions are written in terms of the barred characters $\overline{\chi}_i$ of the transverse $SO(8)$ Lorentz group. The major new notational difference in the heterotic case is that the contributions from the left-moving (internal) worldsheet fermions are now written as products of the unbarred characters χ_i of an internal $SO(16)$ gauge group. Thus, for notational convenience, we are essentially decomposing our $SO(32)$ characters into the characters of the subgroup $SO(16) \times SO(16)$.

Let us now consider the extrapolation of this theory to finite temperature. If we were to apply the standard prescriptions, we would obtain a nine-dimensional extrapolation with partition function

$$
\begin{aligned}
Z \;=\; Z^{(7)}_{\rm boson} \times \{ \quad & \mathcal{E}_0 \;\; \overline{\chi}_V \; (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \\
- \; & \mathcal{E}_{1/2} \;\; \overline{\chi}_S \; (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \\
- \; & \mathcal{O}_0 \;\; \overline{\chi}_C \; (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \\
+ \; & \mathcal{O}_{1/2} \;\; \overline{\chi}_I \; (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \; \} \;.
\end{aligned}
\tag{20}
$$

This is indeed modular invariant, and incorporates proper thermal spin-statistics relations for states with zero thermal windings (*i.e.*, states multiplying the $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ thermal functions). Despite these successes, it is easy to demonstrate that Eq. (20) cannot represent a self-consistent thermal extrapolation according to the construction procedure we laid out in Sect. I — *i.e.*, that this cannot be the partition function of a *bona-fide* self-consistent nine-dimensional string model.

Before giving a definitive argument to this effect, let us begin by noting that there are certain immediate clues that all is not well. First, we observe that Eq. (20) would appear to represent a non-supersymmetric interpolation between two *supersymmetric* limits, one at $T = 0$ and the other at $T \to \infty$, both of which represent the same $SO(32)$ heterotic string model but with opposite spacetime chiralities! Indeed, while the $T \to 0$ limit of Eq. (20) reproduces Eq. (19), the $T \to \infty$ limit yields

$$Z'_{SO(32)} \;=\; Z^{(8)}_{\rm boson} \, (\overline{\chi}_V - \overline{\chi}_C) \, (\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2) \;. \tag{21}$$

This is precisely the same model with which we started at zero temperature, only now involving spacetime spinors of opposite chirality. This is quite unlike the case of the Type II string, where the $T \to \infty$ endpoint model was the non-supersymmetric Type 0A or 0B theory.

Second, we observe that in Eq. (20), the string worldsheet CFT ground state character $\overline{\chi}_I \chi_I^2$ appears within the sector multiplied by $\mathcal{O}_{1/2}$. To see why this is a problem, let us first observe that it follows from modular invariance that this is the only place such a term could possibly have appeared: since the ground state of the heterotic string is not level-matched, having worldsheet energies $(H_R, H_L) = (-1/2, -1)$, invariance under $\tau \to \tau + 1$ forces such a term to be multiplied by the function $\mathcal{O}_{1/2}$, which also fails to be level-matched. In other words, modular invariance requires a term such as $\overline{\chi}_I \chi_I^2$, if it exists, to appear multiplied by $\mathcal{O}_{1/2}$ rather than by any of the other thermal functions. However, this is a problem because the $\mathcal{O}_{1/2}$ sector must be interpreted as completely *twisted* with respect to the $\mathbb{Z}_2$ thermal orbifold. Indeed, no matter whether we run our interpolations from $T \to 0$ to $T \to \infty$ or backwards from $T \to \infty$ to $T \to 0$, the contributions multiplying $\mathcal{O}_{1/2}$ can only correspond to twisted sectors. However, we do not expect to see the ground state of a self-consistent conformal field theory emerging from a twisted sector. Equivalently stated, we expect a term of the form $\overline{\chi}_I \chi_I^2$ to appear multiplied by $\mathcal{E}_0$, $\mathcal{E}_{1/2}$, or $\mathcal{O}_0$, but never $\mathcal{O}_{1/2}$. Thus, combining these observations, we see that a self-consistent heterotic thermal extrapolation should not have a term of the form $\overline{\chi}_I \chi_I^2$ appearing *anywhere* in its partition function. Yet, this term appears within Eq. (20).

In order to diagnose the source of the problem, let us return to our original observation that the nine-dimensional "model" in Eq. (20) seems to interpolate between to supersymmetric endpoints: the $SO(32)$ heterotic string at zero temperature, and the chirality-flipped $SO(32)$ heterotic string at infinite temperature. However, according to our discussion in Sect. I, this can represent a consistent nine-dimensional interpolation only if the chirality-flipped $SO(32)$ model can be viewed as a $\mathbb{Z}_2$ orbifold of the unflipped $SO(32)$ model. We shall now demonstrate that there is no such $\mathbb{Z}_2$ orbifold which can accomplish this transformation.

To see this, let us consider the worldsheet sector giving rise to the gravitino of the original supersymmetric $SO(32)$ model. Recall that in the heterotic string, the gravitino state is realized in the Ramond sector as the spin-3/2 component within the tensor product

$$\text{gravitino:} \qquad \tilde{g}^{\alpha\nu} \; \subset \; \{\tilde{b}_0\}^{\alpha}|0\rangle_R \; \otimes \; \alpha^{\nu}_{-1}|0\rangle_L \; . \tag{22}$$

Here α^{ν}_{-1} denotes the lowest excitation of the left-moving worldsheet coordinate boson X^{ν}, with its Lorentz vector index ν, while $\{\tilde{b}_0\}^{\alpha}$ schematically indicates the Ramond zero-mode combinations which collectively give rise to the spacetime Lorentz spinor index α, as required for the spin-3/2 gravitino state. Note that in order for such a state to be massless and level-matched, it must emerge from a sector in which the left-moving (conformal) side of the heterotic string is in the completely Neveu-Schwarz ground state, while the right-moving (superconformal) side of the heterotic string is in the completely Ramond ground state. Note that in ten dimensions, this is the *only* sector which can ever give rise to spacetime gravitinos, and as such this sector is unique.

So what must our desired orbifold do? In general, orbifolds project certain states out of the spectrum from untwisted sectors, but then introduce new twisted sectors from which additional states may emerge. In our case, the desired orbifold must project out the gravitino that previously emerged from the gravitino sector. This is because the Q-orbifold must break spacetime supersymmetry (or alternatively, because the gravitino has the wrong chirality). However, our orbifolding procedure must also somehow restore an opposite-chirality gravitino from a twisted sector. This is necessary so that the net result of the orbifolding procedure can be the chirality-flipped supersymmetric $SO(32)$ model.

Ordinarily, there are many instances in which a given state might be projected out of the spectrum from an untwisted sector only to re-emerge from a twisted sector. However, as we have noted above, the gravitino sector is *unique* within the context of ten-dimensional heterotic strings — this is the only sector which can provide gravitinos of either chirality. It has a unique worldsheet construction, and thus no other sector can re-introduce the needed opposite-chirality gravitino once the original gravitino has been projected out of the original untwisted sector. We conclude, then, that there is no self-consistent $\mathbb{Z}_2$ orbifold Q which can possibly transform the ten-dimensional supersymmetric $SO(32)$ heterotic string into a chirality-flipped version of itself. It then follows from the construction presented in Sect. I that there is no self-consistent nine-dimensional interpolation between these two ten-dimensional endpoints.

Note that we are not saying that an orbifold cannot project out certain states from an untwisted sector, only to have them re-emerge (even with chirality flips) from a twisted sector. This indeed happens quite often. Rather, we are claiming that for the gravitino in ten dimensions, the actual *worldsheet sector* from which such a state can arise is unique, and thus such a sector cannot be both untwisted and twisted with respect to the same orbifold. In other words, each sector can contribute only once in a given string model.

Note that in this discussion, we are identifying our endpoint models M_1 and M_2 as the supersymmetric $SO(32)$ heterotic string models with opposite chiralities, in accordance with the results of Sect. I. Since $M_1 = \tilde{M}_2$ in this case (where $\tilde{M}_2$ is the T-dual of M_2), one might instead try to form the desired thermal model by interpolating between M_1 and itself, *i.e.*, between M_1 and $\tilde{M}_2$, thereby avoiding the chirality flip. However, the chirality flip is not the real issue here. Indeed, the original $SO(32)$ heterotic gravitino must always be projected out of the spectrum by the Q-orbifold; otherwise, supersymmetry would not be broken by thermal effects in such an interpolation. A new gravitino cannot then be re-introduced from a twisted sector, regardless of its chirality.

We conclude, then, that Eq. (20), although modular invariant, fails to represent a self-consistent thermal extrapolation of the ten-dimensional $SO(32)$ heterotic string. In particular, it cannot correspond to a self-consistent nine-dimensional interpolating string model at the worldsheet level. Identical arguments apply as well to the $E_8 \times E_8$ heterotic string.

IV.   CORRECT FINITE-TEMPERATURE EXTRAPOLATION FOR THE $SO(32)$ HETEROTIC STRING

In order to derive the correct finite-temperature extrapolation for the $SO(32)$ heterotic string, we follow our procedure in Sect. I. Specifically, we must choose an appropriate orbifold Q of the $SO(32)$ string, and then develop the corresponding nine-dimensional interpolating model.

What are the possible self-consistent $\mathbb{Z}_2$ orbifolds of the $SO(32)$ heterotic string? Clearly, this question boils down to the question of identifying possible ten-dimensional $T \to \infty$ endpoints for our corresponding nine-dimensional interpolation. Fortunately, all heterotic strings in ten dimensions have been classified [5]. In addition to the supersymmetric $SO(32)$ and $E_8 \times E_8$ heterotic strings, there are seven additional non-supersymmetric strings. These are the tachyon-free $SO(16) \times SO(16)$ string model as well as six tachyonic string models with gauge groups $SO(32)$, $SO(8) \times SO(24)$, $U(16)$, $SO(16) \times E_8$, $(E_7)^2 \times SU(2)^2$, and E_8. The tachyons in the latter six models all have worldsheet energies $(H_R, H_L) = (-1/2, -1/2)$. However, not all of these models can be realized as $\mathbb{Z}_2$ orbifolds of

the supersymmetric $SO(32)$ model, and even in the remaining cases, not all of the resulting interpolating models will have a radius of compactification that can be interpreted thermally in the field-theory limit, as required by our third condition in Sect. II.

Fortunately, one interpolation meets all of our requirements. Perhaps not surprisingly, this is the interpolation between the supersymmetric $SO(32)$ string and the non-supersymmetric $SO(32)$ string. Note that the non-supersymmetric $SO(32)$ heterotic string has partition function

$$Z = Z_{\text{boson}}^{(8)} \times \left\{ \overline{\chi}_I \left(\chi_I \chi_V + \chi_V \chi_I \right) + \overline{\chi}_V \left(\chi_I^2 + \chi_V^2 \right) - \overline{\chi}_S \left(\chi_S^2 + \chi_C^2 \right) - \overline{\chi}_C \left(\chi_S \chi_C + \chi_C \chi_S \right) \right\}. \tag{23}$$

Following the procedure outlined in Sect. I, we then find that the corresponding nine-dimensional interpolating model has the partition function [6]

$$
\begin{aligned}
Z_{\text{string}}(\tau, T) = Z_{\text{boson}}^{(7)} \times \{ \quad & \mathcal{E}_0 \quad [\overline{\chi}_V \left(\chi_I^2 + \chi_V^2 \right) - \overline{\chi}_S \left(\chi_S^2 + \chi_C^2 \right)] \\
+ \; & \mathcal{E}_{1/2} \quad [\overline{\chi}_V \left(\chi_S^2 + \chi_C^2 \right) - \overline{\chi}_S \left(\chi_I^2 + \chi_V^2 \right)] \\
+ \; & \mathcal{O}_0 \quad [\overline{\chi}_I \left(\chi_I \chi_V + \chi_V \chi_I \right) - \overline{\chi}_C \left(\chi_S \chi_C + \chi_C \chi_S \right)] \\
+ \; & \mathcal{O}_{1/2} \quad [\overline{\chi}_I \left(\chi_S \chi_C + \chi_C \chi_S \right) - \overline{\chi}_C \left(\chi_I \chi_V + \chi_V \chi_I \right)] \; \} .
\end{aligned}
\tag{24}
$$

Note, in particular, that this correctly reproduces Eq. (19) in the $T \to 0$ limit as well as Eq. (23) in the $T \to \infty$ limit. Moreover, it satisfies thermal spin-statistics relations for the massless states with zero string windings: all massless states multiplying $\mathcal{E}_0$ are spacetime bosons, while all massless states multiplying $\mathcal{E}_{1/2}$ are spacetime fermions. (Note in this context there are no massless states which contribute to terms of the forms $\overline{\chi}_{S,V} \chi_{S,C}^2$, since $\chi_{S,C}^2$ has conformal dimension $h = 2$.)

We stress that this is the unique thermal extrapolation which satisfies the conditions we put forth in Sect. II. Indeed, only this extrapolation corresponds to a self-consistent nine-dimensional interpolating model with an identifiable thermal radius of compactification with proper thermal spin-statistics relations in the field-theory limit. However, there are some unique features involved in such an interpolation. While it was perhaps already expected from Ref. [4] that states with non-trivial thermal winding modes might behave in a counter-intuitive fashion, violating finite-temperature spin-statistics relations in the $\mathcal{O}_0$ and $\mathcal{O}_{1/2}$ sectors, we now see that our interpolations necessarily have apparent Planck-scale thermal spin-statistics violations even for the states with *zero* windings, *i.e.*, states with conformal dimensions $h > 1$ which appear in the $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ sectors. Planck-scale violations of this sort appear to be unavoidable, even for zero-winding states, and (as we shall argue below) are required by modular invariance in the context of self-consistent interpolating models. It would be interesting to understand the thermal implications of these states as far as Planck-scale physics is concerned.

Note, however, that although these Planck-scale states appear to violate *thermal* spin-statistics relations, they still obey zero-temperature spin-statistics relations, as required. In other words, all spacetime bosons contribute positively to the partition function, while all spacetime fermions contribute negatively, with minus signs in front of their corresponding characters.

V. IMPLICATIONS FOR THE HAGEDORN TRANSITION

Let us now discuss the implications of these results for the Hagedorn transition [1, 4, 7]. Our focus here will be on the tachyons and temperature associated with the Hagedorn transition; for a more complete set of references concerning the history and possible interpretations and implications of the Hagedorn transition, see Ref. [8].

A. The Hagedorn transition: UV versus IR

We begin with several preliminary remarks concerning the UV/IR nature of the Hagedorn transition. In general, once we have determined the correct finite-temperature partition function $Z_{\text{string}}(\tau, T)$ for a given zero-temperature string model, the one-loop thermal vacuum amplitude $\mathcal{V}(T)$ (the analogue of the logarithm of the statistical-mechanical partition function) is given by a modular integral of the form [2]

$$\mathcal{V}(T) \equiv -\tfrac{1}{2} \mathcal{M}^{D-1} \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im} \, \tau)^2} \, Z_{\text{string}}(\tau, T) \tag{25}$$

where $\mathcal{M} \equiv M_{\text{string}}/(2\pi)$ is the reduced string scale; D is the number of non-compact spacetime dimensions; and where

$$\mathcal{F} \equiv \{\tau : \ |\text{Re}\,\tau| \leq \tfrac{1}{2}, \text{Im}\,\tau > 0, |\tau| \geq 1\} \qquad (26)$$

is the fundamental domain of the modular group. We shall often abbreviate $\tau_1 \equiv \text{Re}\,\tau$ and $\tau_2 \equiv \text{Im}\,\tau$. Given this definition for $\mathcal{V}$, the free energy F, internal energy U, entropy S, and specific heat c_V then follow from the standard thermodynamic definitions $F \equiv T\mathcal{V}$, $U \equiv -T^2 d\mathcal{V}/dT$, $S \equiv -dF/dT$, and $c_V \equiv dU/dT$.

In string theory, the Hagedorn transition is usually associated with a divergence or other discontinuity in the vacuum amplitude $\mathcal{V}(T)$ as a function of temperature. There are two ways in which such a divergence may arise. First, of course, there may be an ultraviolet divergence due to the well-known exponential rise in the degeneracy of string states. However, such an ultraviolet divergence would normally be associated with the $\tau_2 \to 0$ region of integration in Eq. (25), and we see from Eq. (26) that this region of integration has been eliminated from the integral as a result of modular invariance — *i.e.*, as the result of the truncation of the region of integration to the modular group fundamental domain in Eq. (26). Thus, strictly speaking, there can be no UV divergence contributing to $\mathcal{V}(T)$. On the other hand, there may be purely infrared divergences coming from on-shell physical tachyons within $Z_{\text{string}}(\tau, T)$; such states would lead to divergences in the infrared region $\tau_2 \to \infty$. Thus, a study of the Hagedorn transition in string theory essentially reduces to a study of the possible tachyonic structure of $Z_{\text{string}}(\tau, T)$ as a function of temperature.

Before proceeding further, we caution that we reach this conclusion only because we have chosen to work in the so-called $\mathcal{F}$-representation for $\mathcal{V}(T)$ given in Eq. (25). By contrast, utilizing Poisson resummations and modular transformations [9], we can always rewrite $\mathcal{V}(T)$ as the integration of a different integrand $Z'_{\text{string}}(\tau, T)$ over the strip

$$\mathcal{S} \equiv \{\tau : \ |\text{Re}\,\tau| \leq \tfrac{1}{2}, \text{Im}\,\tau > 0\} \ . \qquad (27)$$

In such an $\mathcal{S}$-representation, the former IR divergence as $\tau_2 \to \infty$ is transformed into a UV divergence as $\tau_2 \to 0$. This formulation thus has the advantage of relating the Hagedorn transformation directly to a UV phenomenon such as the exponential rise in the degeneracy of states. However, both formulations are mathematically equivalent; indeed, modular invariance provides a tight relation between the tachyonic structure of a given partition function and the rate of exponential growth in its asymptotic degeneracy of states [10, 11, 12]. In the following, therefore, we shall utilize the $\mathcal{F}$-representation for $\mathcal{V}(T)$ and focus on only the tachyonic structure of $Z_{\text{string}}(\tau, T)$, but we shall comment on the connection to the asymptotic degeneracy of states in Sect. V C.

B.   A new Hagedorn temperature for heterotic strings

So what then are the potential tachyonic states within $Z_{\text{string}}(\tau, T)$, and at what temperature T_H do they first arise? In other words, at what critical temperature T_H do new massless states emerge within Z_{string}, on their way to becoming tachyonic? Note that we are focusing on *thermally massless* states, *i.e.*, states whose masses depend on temperature, states for which masslessness is achieved at a critical temperature T_H as the result of a balance between a tachyonic non-thermal mass and an additional positive non-zero thermal mass contribution. It is sufficient to focus on such massless states since their emergence is the signal of the long-range order normally associated with a phase transition. These are the states which then presumably become tachyonic beyond T_H, leading to the instabilities normally associated with a phase transition.

Given our results for $Z_{\text{string}}(\tau, T)$ in Sects. III and IV, it is straightforward to obtain the corresponding Hagedorn temperatures. Let us begin by considering the case of the Type II strings, for which the appropriate thermal function is given in Eq. (17). Recalling the conformal dimensions associated with the different characters in Eq. (17), we see that the only potentially tachyonic contributions in this expression arise from the term $\overline{\chi}_I \chi_I \mathcal{O}_0$. Thus, only this sector has the potential to give rise to thermally massless level-matched states. Solving the conditions for masslessness, we find that the $(m, n) = (0, \pm 1)$ thermal excitations of the $(H_R, H_L) = (-1/2, -1/2)$ tachyons within $\overline{\chi}_I \chi_I$ will become thermally massless at the temperature $T_H = \mathcal{M}/\sqrt{2}$. These thermal states are massive for $T < T_H$, and tachyonic for $T > T_H$. We thus identify $T_H = \mathcal{M}/\sqrt{2}$ as the Hagedorn temperature for Type II strings. Note that this analysis is in complete agreement with the standard derivations [1, 4] of the Hagedorn temperature for Type II strings.

However, the main difference occurs in the case of the heterotic string. Performing exactly the same analysis for the thermal partition function given in Eq. (24), we find that only the term $\overline{\chi}_I(\chi_I \chi_V + \chi_V \chi_I)\mathcal{O}_0$ is capable of giving rise to thermally massless level-matched states. Indeed, the $SO(16) \times SO(16)$ character $(\chi_I \chi_V + \chi_V \chi_I)$ gives rise to the 32 on-shell $(H_R, H_L) = (-1/2, -1/2)$ tachyons of non-supersymmetric $SO(32)$ string which serves as the $T \to \infty$ endpoint of the interpolation, and we find that the $(m, n) = (0, \pm 1)$ thermal excitations of these states are massless at $T_H = \mathcal{M}/\sqrt{2}$, massive below this temperature, and tachyonic above it. This is exactly the same as for the Type II string, and there are no other tachyonic sectors within Eq. (24) which could give rise to other phase transitions at lower

temperatures. We thus conclude that the ten-dimensional supersymmetric $SO(32)$ heterotic string has a Hagedorn temperature given by $T_H = \mathcal{M}/\sqrt{2}$, which is exactly the same as the Hagedorn temperature for the Type II string. A similar analysis with the same result also applies for the $E_8 \times E_8$ heterotic string as well as the non-supersymmetric tachyon-free $SO(16) \times SO(16)$ string. We thus find that

$$T_H \;=\; \frac{\mathcal{M}}{\sqrt{2}} \;=\; \frac{M_{\text{string}}}{2\sqrt{2}\pi} \qquad \text{for } all \text{ tachyon-free closed strings in } D = 10 , \tag{28}$$

both Type II and heterotic! In other words, by carefully constructing self-consistent interpolating models with their required GSO projections, we have uncovered a universal Hagedorn temperature for all closed tachyon-free strings in ten dimensions.

This is clearly a major difference relative to our usual expectations. Indeed, if we had performed the same analysis using the (inconsistent) expression in Eq. (20), we would have found that the $(H_R, H_L) = (-1/2, -1)$ off-shell tachyon within the sector $\overline{\chi}_I \chi_I^2 \mathcal{O}_{1/2}$ contains thermal excitations $(m, n) = \pm(1/2, 1)$ which become thermally massless at the expected (heterotic) Hagedorn temperature $T_H = 2\mathcal{M}/(2 + \sqrt{2}) = (2 - \sqrt{2})\mathcal{M}$. However, as we discussed in Sect. IV, this state is actually GSO-projected out of the spectrum when we construct the proper thermal interpolating model in Eq. (24). We thus find that the Hagedorn temperature for the heterotic string is altered.

It is not surprising, perhaps, that the Type II and heterotic strings share a common Hagedorn temperature once the correct thermal extrapolations are taken into account. In the case of the Type II string, the ground state is a tachyon with worldsheet energies $(H_R, H_L) = (-1/2, -1/2)$. This state is level-matched, and survives into the corresponding thermal extrapolating function in Eq. (17). In the case of the heterotic string, by contrast, the ground state has vacuum energies $(H_R, H_L) = (-1/2, -1)$. Although this would naively appear to change the associated Hagedorn temperature, the fact that this state is not level-matched, together with modular invariance, ends up forcing this state to appear within the thermally twisted sector $\mathcal{O}_{1/2}$ where its appearance would be inconsistent. Thus, all thermal contributions from this state are projected out, and only the "next-deepest" tachyon, again with $(H_R, H_L) = (-1/2, -1/2)$, survives to affect the resulting thermodynamics. Since this surviving heterotic tachyon has exactly the same worldsheet energies as the Type II ground state, the heterotic and Type II theories have exactly the same Hagedorn temperatures.

C. Reconciling the new Hagedorn temperature with the asymptotic degeneracy of states

The above arguments are clearly based on an IR analysis of the tachyonic structure of our thermal interpolating models. One may therefore wonder how it is possible to find a Hagedorn temperature $T_H = \mathcal{M}/\sqrt{2}$ for a heterotic string such as the $SO(32)$ string, given that its zero-temperature bosonic and fermionic densities of states nevertheless continue to exhibit an exponential rate of growth normally associated with the usual heterotic Hagedorn temperature $T_H = (2 - \sqrt{2})\mathcal{M}$. This is a very important question which we shall now address.

We shall develop our response in several layers. First, let us recall the usual direct connection between the asymptotic degeneracy of states and the corresponding Hagedorn temperature [7]: if g_M denotes the number of states with mass M, then the thermal partition function is given by $Z(T) = \sum g_M e^{-M/T}$. However, if $g_M \sim e^{\alpha M}$ as $M \to \infty$, then $Z(T)$ diverges for $T \geq T_H \equiv 1/\alpha$. This appears to provide a firm link between the Hagedorn temperature and the asymptotic degeneracy of states.

Of course, one might argue that this kind of partition function $Z(T) = \sum g_M e^{-M/T}$ is not a proper string-theoretic partition function; it assumes that the string is nothing but a collection of the states to which its excitations give rise. Indeed, we must perform a proper string-theoretic vacuum-amplitude calculation as outlined in Eq. (25), using a string partition function $Z_{\text{string}}(\tau, T)$ which depends not only on the temperature T but also a torus parameter τ. We must then integrate over τ over a restricted fundamental domain.

However, the same basic argument connecting the asymptotic degeneracy of states with the Hagedorn transition continues to apply, even for the proper string-theory calculation. After all, we may easily transform to the $\mathcal{S}$-representation for $\mathcal{V}(T)$, as discussed in Sect. V A; in this representation, the connection between the asymptotic rise in the degeneracy of states and the UV divergence as $\tau_2 \to 0$ becomes manifest in the $\tau_2 \to 0$ region. How then can we interpret the increase in the Hagedorn temperature from the traditional heterotic value $T_H = (2 - \sqrt{2})\mathcal{M}$ to the new, slightly higher value $T_H = \mathcal{M}/\sqrt{2}$? It seems that this would require a corresponding *decrease* in the exponential rate of growth in the asymptotic density of string states.

To answer this question, we must first recognize that the transition from the $\mathcal{F}$-representation to the $\mathcal{S}$-representation is highly non-trivial in the case of string theories containing spacetime fermions (such as the Type II and heterotic strings). In the case of the bosonic string, for example, the thermal partition function necessarily takes the factorized form given in Eq. (1); the absence of spacetime fermions implies that no subsequent thermal $\mathbb{Z}_2$ orbifolding is required.

Such a partition function is particularly easy to transform to the $\mathcal{S}$-representation where the connection between the degeneracy of states and the Hagedorn temperature is immediate and apparent in the $\tau_2 \to 0$ region (which is why we do not find a change in the Hagedorn temperature for bosonic strings, assuming they could be made stable at zero temperature). However, as we have seen above, for Type II and heterotic strings the thermal partition function necessarily takes the more complicated form given in Eq. (11). The failure of this form to factorize — indeed, the presence of the half-integer shifts in the thermal momenta for the $\mathcal{E}_{1/2}$ and $\mathcal{O}_{1/2}$ sectors — is the direct consequence of the $\mathbb{Z}_2$ thermal orbifold. However, when we take modular transformations of this partition function in order to shift to the $\mathcal{S}$-representation for $\mathcal{V}(T)$, as described in Ref. [9], this half-integer shift is transformed into non-trivial $\mathbb{Z}_2$ *phases* (*i.e.*, minus signs) in the corresponding asymptotic degeneracies of states. These minus signs act to cancel the dominant exponential divergences in the degeneracies of states, allowing *subleading* exponential divergences to dominate. (Such subleading exponential terms are discussed fully in Refs. [10, 11, 12].) This subleading, reduced asymptotic growth then correlates directly with the increase in the Hagedorn temperature that we have found.

Still, one may argue on general conformal-field-theory (CFT) grounds that such a change in the Hagedorn temperature should not be possible. After all, the heterotic string has a worldsheet CFT with central charges $(c_R, c_L) = (12, 24)$ in light-cone gauge; this is implicit in the fact that the heterotic string ground state has worldsheet vacuum energies $(H_R, H_L) = (-c_R/24, -c_L/24) = (-1/2, -1)$. The usual arguments for the Hagedorn transition then lead directly to a Hagedorn temperature given in terms of these central charges as

$$T_H \;=\; \left(\sqrt{\frac{c_L}{24}} + \sqrt{\frac{c_R}{24}} \right)^{-1} \mathcal{M} \;, \tag{29}$$

and we have seen that this argument certainly holds in the case of the ten-dimensional Type II strings. However, this argument fails in the case of the heterotic strings because the heterotic string ground state, encapsulated within the CFT character $\overline{\chi}_I \chi_I^2$, has been *GSO-projected* out of the thermal spectrum in Eq. (24). Indeed, as stressed above, this is the primary difference between Eq. (20) and Eq. (24). Or, to phrase this point slightly differently, the GSO-projections inherent in Eq. (24) have deformed the worldsheet CFT of the theory so that it effectively has a new string ground state consisting of the $(H_R, H_L) = (-1/2, -1/2)$ tachyons encapsulated within the surviving term $\overline{\chi}_I (\chi_I \chi_V + \chi_V \chi_I)$ in Eq. (24). Since this effective ground state has depth $(H_R, H_L) = (-1/2, -1/2)$, we can identify the "effective" central charge of this deformed left/right worldsheet theory after the GSO projections as $(c_R, c_L) = (12, 12)$, just as for the Type II string.

It is also easy to understand how this deformation manages to affect the asymptotic exponential growth in the degeneracy of states within Z_{therm}. Recall that in a given CFT with central charge c, each character $\chi_i(\tau)$ represents a trace over the Verma module associated with the corresponding primary field ϕ_i, and has a q-expansion of the form [10, 11, 12]

$$\chi_i(\tau) \;=\; q^{h_i - c/24} \sum_{p=0}^{\infty} a_p^{(i)} \, q^p \qquad \text{with} \;\; a_p^{(i)} \;\sim\; \exp\left(4\pi \sqrt{\frac{cp}{24}} \right) \quad \text{as } p \to \infty \;. \tag{30}$$

Here h_i is the conformal weight of the primary field ϕ_i. In deriving this result via the standard contour-integral methods [10, 11, 12], one uses the fact that each character χ_i is connected to the identity character χ_I of the CFT ground state through a $\tau \to -1/\tau$ modular transformation. (The existence of this connection is guaranteed from the CFT fusion rules and the Verlinde formula [13].) It is for this reason that the q-expansion of each character χ_i of the CFT *individually* has coefficients which exhibit an exponential growth rate related to the underlying central charge of the worldsheet CFT. Likewise, this is why *individual* products of left- and right-moving characters yield asymptotic growth rates consistent with the traditional Hagedorn temperature.

However, in our thermal partition function in Eq. (24), we see that we essentially have four *combinations* of left/right characters which multiply our $\mathcal{E}/\mathcal{O}$ functions. Explicitly, these combinations are given by

$$\begin{aligned} Z^{(1)} &= \overline{\chi}_V \, (\chi_I^2 + \chi_V^2) \;-\; \overline{\chi}_S \, (\chi_S^2 + \chi_C^2) \\ Z^{(2)} &= \overline{\chi}_V \, (\chi_S^2 + \chi_C^2) \;-\; \overline{\chi}_S \, (\chi_I^2 + \chi_V^2) \\ Z^{(3)} &= \overline{\chi}_I \, (\chi_I \chi_V + \chi_V \chi_I) \;-\; \overline{\chi}_C \, (\chi_S \chi_C + \chi_C \chi_S) \\ Z^{(4)} &= \overline{\chi}_I \, (\chi_S \chi_C + \chi_C \chi_S) \;-\; \overline{\chi}_C \, (\chi_I \chi_V + \chi_V \chi_I) \;. \end{aligned} \tag{31}$$

These four left/right character combinations close amongst themselves under modular transformations, and thus function as a new "effective" set of characters for our "effective" (deformed) left/right worldsheet CFT. However, this effective character set does not contain the heterotic CFT ground-state $\overline{\chi}_I \chi_I^2$; indeed, the most tachyonic term that survives in this character set is the term $\overline{\chi}_I (\chi_I \chi_V + \chi_V \chi_I)$ within $Z^{(3)}$. We thus see that this effective left/right

CFT has a reduced "effective" central charge compared with the original left/right CFT prior to GSO projections, as discussed above.

This is also directly evident from the $(q, \overline{q})$-expansions of the left/right character combinations $Z^{(1)}$ and $Z^{(2)}$. For example, looking at $Z^{(1)}$, we find that the first term $\overline{\chi}_V(\chi_I^2 + \chi_V^2)$ individually has a $(q, \overline{q})$-expansion with coefficients (mass-level degeneracies) exhibiting an exponential growth rate consistent with the traditional Hagedorn temperature. However, the same is also true of the second term within $Z^{(1)}$, namely $\overline{\chi}_S(\chi_S^2 + \chi_C^2)$, and the fact that these terms are *subtracted* in forming $Z^{(1)}$ actually ends up *cancelling* this leading exponential behavior. What remains is only a subleading exponential growth rate consistent with our re-identified Hagedorn temperature. A similar cancellation also holds for $Z^{(2)}$; note that cancellations of these sorts are discussed in detail in Ref. [12]. Thus, at the partition-function level, this reduction in the Hagedorn temperature is a direct result of the minus signs within the combinations $Z^{(1)}$ and $Z^{(2)}$ in Eq. (31), which in turn are a direct result of the non-standard thermal spin-statistics relations that we have already observed at the Planck scale.

In some sense, this entire argument may be summarized as follows. Let us look again at the original partition function of the $SO(32)$ heterotic string model in Eq. (19). As a result of spacetime supersymmetry, this partition function vanishes identically — *i.e.*, all of its level-degeneracy coefficients are identically zero. There is no exponential growth here at all. But one does not examine the *whole* partition function in order to derive a Hagedorn temperature; one instead looks at its separate bosonic and fermionic contributions. Ordinarily, these contributions would be identified as

$$Z_{SO(32)}^{(\text{bosonic})} = Z_{\text{boson}}^{(8)} \, \overline{\chi}_V \left(\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2 \right) , \qquad Z_{SO(32)}^{(\text{fermionic})} = Z_{\text{boson}}^{(8)} \, \overline{\chi}_S \left(\chi_I^2 + \chi_V^2 + \chi_S^2 + \chi_C^2 \right) , \qquad (32)$$

and indeed each of these expressions separately exhibits an exponential rise in the degeneracy of states which is consistent with the traditional heterotic Hagedorn temperature. But what do we really mean by "bosonic" and "fermionic"? Taking a thermodynamic definition, we are forced to identify such states according to their periodicities around the thermal circle. Therefore, given our $(D-1)$-dimensional interpolating-model analysis, we now see that for the heterotic string, Eq. (32) is not the correct way to separate the total partition function into its separate thermal components. Instead, for thermal purposes, we now see that the proper separation is into the different components

$$
\begin{aligned}
Z_{SO(32)}^{(\text{"bosonic"})} &= Z_{\text{boson}}^{(8)} \left[\overline{\chi}_V \left(\chi_I^2 + \chi_V^2 \right) \, - \, \overline{\chi}_S \left(\chi_S^2 + \chi_C^2 \right) \right] , \\
Z_{SO(32)}^{(\text{"fermionic"})} &= Z_{\text{boson}}^{(8)} \left[\overline{\chi}_S \left(\chi_I^2 + \chi_V^2 \right) \, - \, \overline{\chi}_V \left(\chi_S^2 + \chi_C^2 \right) \right] ,
\end{aligned}
\qquad (33)
$$

since these are the components that appear in the $\mathcal{E}_0$ and $\mathcal{E}_{1/2}$ sectors when the proper thermal extrapolation is constructed. It is therefore *these* components which function as the "bosonic" and "fermionic" contributions as far as thermal effects are concerned, and indeed these are the components which exhibit the slower exponential growth associated with our re-identified Hagedorn temperature. We stress that both Eq. (32) and Eq. (33) correctly separate those massless bosonic and fermionic states which survive in the field-theory limit. Their only difference is in their treatment of the stringy Planck-scale states which have no field-theoretic limits. In other words, Eq. (33) correctly identifies bosons and fermions according to their thermal behaviors; what is unusual is the connection between this behavior and the spacetime Lorentz spins of the Planck-scale states.

We thus conclude that all tachyon-free closed strings in ten dimensions share a universal Hagedorn temperature. Although the heterotic string would naively appear to have a slightly lower Hagedorn temperature than the Type II string due to its non-level-matched ground state, self-consistency also requires a set of non-trivial GSO projections which compensate for the non-level-matched ground state by inducing a cancellation in the asymptotic degeneracies of states, thereby pushing the associated Hagedorn temperature back to the Type II value.

VI. BEYOND TEN DIMENSIONS: ADDITIONAL GENERAL OBSERVATIONS

In ten dimensions, we found that each of the closed string theories which are tachyon-free at zero temperature has a Hagedorn transition associated with a tachyon that emerges in its thermal extrapolation, with worldsheet energies $(H_R, H_L) = (-1/2, -1/2)$. For the supersymmetric Type II strings, we have seen that this tachyon always emerges in the thermal extrapolation because the only possible $T \to \infty$ endpoints for the corresponding interpolating models are the Type 0 strings, which necessarily contain these tachyons. In the case of the $SO(32)$ and $E_8 \times E_8$ heterotic strings, we have seen that we must also identify the appropriate non-supersymmetric heterotic string models to serve as suitable $T \to \infty$ endpoints. While the tachyon-free $SO(16) \times SO(16)$ string could have logically served as this endpoint, it turns out that this choice would violate thermal spin-statistics constraints in both the $SO(32)$ and $E_8 \times E_8$ cases [6, 14]. Thus, in each case, our $T \to \infty$ endpoint model must be one of the remaining non-supersymmetric string

models [*e.g.*, for the $SO(32)$ string, we found that the endpoint was the non-supersymmetric $SO(32)$ string]. Since each of these remaining non-supersymmetric models has tachyons with $(H_R, H_L) = (-1/2, -1/2)$, we again have a Hagedorn transition, albeit with a re-identified Hagedorn temperature.

Given these results, two obvious questions arise. First, is it a general property that *all* heterotic strings will have new, re-identified Hagedorn temperatures? Our belief is that this is indeed the case, regardless of the spacetime dimension. As we have argued in Sect. V, the usual Hagedorn transition in the heterotic case requires the existence of the term $\overline{\chi}_I \chi_I^2 \mathcal{O}_{1/2}$ (or more generally, the ground-state character multiplied by $\mathcal{O}_{1/2}$) in the corresponding thermal interpolating partition function, yet the thermally twisted nature of the $\mathcal{O}_{1/2}$ sector precludes this from happening. We believe that this is a general argument which transcends the particular gravitino-based orbifold argument which was also provided in Sect. III.

A second, equally important issue concerns whether it might be possible to eliminate the Hagedorn phase transition completely by finding a zero-temperature string model for which the appropriate thermal extrapolation involves a $T \to \infty$ endpoint model which is *non-supersymmetric but tachyon-free*. While this did not occur in ten dimensions, this remains a logical possibility in lower dimensions where many such non-supersymmetric tachyon-free string models exist, both of the superstring and heterotic string variety.

While we do not know the answer to this question in the case of the superstring, we can prove that it is impossible to evade such a phase transition entirely in the cases of heterotic strings which are supersymmetric at zero temperature. Our proof runs as follows [8]. Even if the appropriate $T \to \infty$ endpoint theory happens to be tachyon-free, there will always exist another state in the thermal extrapolation whose thermal excitations will trigger a non-trivial phase transition. This is the so-called "proto-gravitino" state:

$$\text{proto-gravitino:} \qquad \tilde{\phi}^\alpha \;\equiv\; \{\tilde{b}_0\}^\alpha |0\rangle_R \;\otimes\; |0\rangle_L \;. \tag{34}$$

Note that this state is constructed exactly as the gravitino in Eq. (22), but without the left-moving worldsheet coordinate excitation. However, it is important to realize that *GSO projections are completely insensitive to the presence or absence of excitations of the worldsheet coordinate bosonic fields*. Thus, if our zero-temperature heterotic string model is supersymmetric and the gravitino is therefore present in the original zero-temperature theory, then the proto-gravitino must also always be present in the original zero-temperature theory.

Since this state emerges, like the gravitino itself, from the untwisted (fermionic) gravity sector of the original $T \to 0$ model, its contributions must appear multiplied by $\mathcal{E}_{1/2}$ within any self-consistent heterotic thermal extrapolation away from that model. [For example, in the case of the $SO(32)$ heterotic string interpolation in Eq. (24), the proto-gravitino contribution was hiding within the term $-\overline{\chi}_S \chi_I^2 \mathcal{E}_{1/2}$.] However, this proto-gravitino state is then necessarily endowed with a thermal $(m, n) = (1/2, 2)$ excitation which is massive for all $T \neq T_*$ but exactly massless at the single temperature $T = T_*$ where $T_* \equiv 2\mathcal{M}$. (Note that since this state is fermionic, Lorentz invariance prevents it from becoming tachyonic at any temperature.) The sudden appearance of a new massless state at $T = T_*$ signals the emergence of long-range order in the thermal theory, and can again be associated with a Hagedorn-like phase transition. However, since this state hits masslessness only once as a function of temperature and never becomes tachyonic, this turns out [8] to be a very weak, p^{th}-order phase transition, where p is related to the spacetime dimension D:

$$p \;=\; \begin{cases} D & \text{for } D \text{ even} \\ D-1 & \text{for } D \text{ odd} \,. \end{cases} \tag{35}$$

Thus, we conclude that for supersymmetric heterotic strings, it is never possible to completely evade a Hagedorn-like phase transition. However, the phase transition associated with the proto-gravitino state appears only at the relatively high temperature $T_* \equiv 2\mathcal{M}$, and thus will be completely irrelevant if tachyon-induced Hagedorn transitions appear at lower temperatures.

VII. CONCLUSIONS

In this paper, we investigated the manner in which a given zero-temperature string model may be extrapolated to finite temperature. Following relatively conservative conditions for self-consistency, we nevertheless found that the traditional Hagedorn transition does not exist for heterotic strings but is instead replaced by a new, "re-identified" Hagedorn transition which emerges at the somewhat higher temperature normally associated with Type II strings. This allowed us to uncover a universal Hagedorn temperature for all tachyon-free closed string theories in ten dimensions. We also showed that these results are not in conflict with the exponential rise in the degeneracy of string states in these models.

Clearly, many outstanding questions remain. Perhaps the two most critical are the issues of the *existence* and *uniqueness* of thermal extrapolations satisfying the general criteria we put forth in Sect. II. In other words, it

is important to demonstrate that, for any given D-dimensional zero-temperature string model, there always exists one and only one suitable corresponding $T \to \infty$ endpoint D-dimensional string model such that the corresponding $(D-1)$-dimensional interpolation is thermally consistent according to our general criteria, including proper spin-statistics relations. In ten dimensions, we have already seen that such extrapolations exist and are unique. However, neither property has been proven in lower dimensions. This is clearly an important issue that requires further study.

Another interesting question concerns the thermal fate of string models which are non-supersymmetric but tachyon-free at zero temperature: is it ever possible that such a non-supersymmetric model will have a thermal extrapolation whose $T \to \infty$ limit is *supersymmetric*? If so, this would be an example of a situation in which the zero-temperature theory is non-supersymmetric, but in which thermal effects compensate for this inequity between bosons and fermions and thereby *introduce* (rather than break) supersymmetry as $T \to \infty$. In other words, such thermal effects would be "SUSY-making" rather than SUSY-breaking, with SUSY-breaking occurring at *lower* temperatures. This phenomenon would be intrinsically string-theoretic, since only for closed strings does the $T \to \infty$ limit yield a theory of the same dimensionality as the original zero-temperature theory. No examples exhibiting this phenomenon exist in ten dimensions, but it would be interesting to explore whether such examples might exist in lower dimensions.

Acknowledgments

Note Added:

More than three weeks after these results originally appeared in Ref. [8], another article appeared [15] whose author claims that there is no Hagedorn transition whatsoever for bosonic strings, Type II strings, heterotic strings, or even Type I strings. Clearly, we disagree with the results of that paper, and shall limit our comments to version 4 of Ref. [15] (which is the current version as we write this). Although the modular-invariance errors of the previous versions have been corrected, we believe that the relevant thermal expressions given in version 4 for Type II and heterotic strings (Eqs. (23) and (67) of Ref. [15], respectively) are still manifestly inconsistent. One error, for example, concerns thermal spin-statistics: both expressions contain massless bosonic and fermionic states with incorrect modings around the thermal circle. [For example, the first line of Eq. (23) of Ref. [15] necessarily contains contributions from the gravity multiplet, yet these are multiplied by the *sum* $\mathcal{E}_0 + \mathcal{E}_{1/2}$, thereby giving the graviton both periodic *and* anti-periodic thermal excitations around the thermal circle!] Second, Eq. (23) of Ref. [15] does not represent a self-consistent interpolation, since the $T \to \infty$ limit does not correspond to any ten-dimensional Type II string. Third, although the author of Ref. [15] claims to find an *infinite* number of thermal tachyons in the Type II thermal theory starting at $T = 0$, this is because the $T \to 0$ limit of Eq. (23) of Ref. [15] is actually the tachyonic Type 0A/0B theory, not the desired Type IIA/B theory. Indeed, although the author claims that Eq. (23) of Ref. [15] is the *unique* modular-invariant Type II thermal partition function, the correct partition function is actually our Eq. (17), which leads to the usual Hagedorn transition, as expected. Finally, the author of Ref. [15] claims to see no evidence for the heterotic phase transition that we discussed above, induced by the proto-gravitino state. Once again, this is due to the inconsistency of the expressions in Ref. [15]; as we proved in our original paper [8], this state must always exist if the $T \to 0$ heterotic theory is supersymmetric. Indeed, when we fix Eq. (67) of Ref. [15] by copying the results from previous papers [14] correctly and with the correct *supersymmetric* $T \to 0$ limit, we find exactly the behavior that we described above. Indeed, this sort of phase transition has been recently discussed by other authors as well in other contexts (see, *e.g.*, Ref. [16]). Overall, however, the biggest difference between our approach and that of Ref. [15] is that we construct our thermal extrapolations by demanding the physical criteria as we laid out in our Sect. II; we then determine the properties of the Hagedorn transition as a consequence. By contrast, the author of Ref. [15] is implicitly requiring that there be no Hagedorn transition as a *precondition* when demanding that all thermal extrapolations be tachyon-free at all temperatures, regardless of other self-consistency checks.

The purpose of this Note Added has merely been to clarify the primary differences between our work and that of Ref. [15], as these differences have apparently confused several readers.

[1] For an introduction, see:
  M. B. Green, J. A. Schwarz and E. Witten, *Superstring Theory, Vols. I and II* (Cambridge University Press, 1987);
  J. Polchinski, *String Theory, Vols. I and II* (Cambridge University Press, 1998).
[2] J. Polchinski, Commun. Math. Phys. **104**, 37 (1986).
[3] R. Rohm, Nucl. Phys. B **237**, 553 (1984).
[4] J. J. Atick and E. Witten, Nucl. Phys. B **310**, 291 (1988).
[5] H. Kawai, D. C. Lewellen and S. H. H. Tye, Phys. Rev. D **34**, 3794 (1986).
[6] J. D. Blum and K. R. Dienes, Phys. Lett. B **414**, 260 (1997) [arXiv:hep-th/9707148]; Nucl. Phys. B **516**, 83 (1998) [arXiv:hep-th/9707160].
[7] R. Hagedorn, Nuovo Cim. Suppl. **3**, 147 (1965).
[8] K. R. Dienes and M. Lennek, arXiv:hep-th/0505233.
[9] B. McClain and B. D. B. Roth, Commun. Math. Phys. **111**, 539 (1987);
  K. H. O'Brien and C. I. Tan, Phys. Rev. D **36**, 1184 (1987).
[10] G. H. Hardy and S. Ramanujan, Proc. London Math. Soc. **17**, 75 (1918).
[11] I. Kani and C. Vafa, Commun. Math. Phys. **130**, 529 (1990).
[12] K. R. Dienes, Nucl. Phys. B **429**, 533 (1994) [arXiv:hep-th/9402006].
[13] E. P. Verlinde, Nucl. Phys. B **300**, 360 (1988);
  G. W. Moore and N. Seiberg, Phys. Lett. B **212**, 451 (1988).
[14] H. Itoyama and T. R. Taylor, Phys. Lett. B **186**, 129 (1987).
[15] S. Chaudhuri, arXiv:hep-th/0506143.
[16] J. L. Davis, F. Larsen and N. Seiberg, arXiv:hep-th/0505081.